

THE SMOOTH REPRESENTATIONS OF $\mathrm{GL}_2(\mathcal{O})$

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ABSTRACT. We present some unpublished results of Kutzko together with results of Hill, giving a classification of the smooth (complex) representations of $\mathrm{GL}_2(\mathcal{O})$, where \mathcal{O} is the ring of integers in a local field with finite residue field.

1. INTRODUCTION

Let \mathcal{O} be the ring of integers in a local field with finite residue field, and let \mathfrak{p} be its maximal ideal. The smooth representations of $\mathrm{GL}_2(\mathcal{O})$, or equivalently, the representations of the finite groups $\mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^r)$, $r \geq 1$, have in one form or another been known to some mathematicians since the late 70s. There has been two different approaches to this problem. On the one hand, there is the Weil representation approach of Nobs and Wolfart (applied to the case $\mathcal{O} = \mathbb{Z}_p$ in [13]). On the other hand, there is the approach via orbits and Clifford theory due to Kutzko (unpublished), and independently to Nagornyj [11].

The related case of $\mathrm{SL}_2(\mathbb{Z}_p)$, $p \neq 2$, has been studied by Kloosterman [9, 10], Tanaka [17, 18], Kutzko (unpublished), and Shalika (for general \mathcal{O} and $p \neq 2$) [15]. Another description of the representations of $\mathrm{SL}_2(\mathbb{Z}_p)$ (including the case $p = 2$) was given by Nobs and Wolfart [12, 14], using Weil representations. The case $\mathrm{PGL}_2(\mathcal{O})$ with odd residue characteristic has been treated by Silberger [16].

The series of papers by Hill [4, 5, 6, 7] use the method of orbits and Clifford theory as a basis for dealing with the general case of $\mathrm{GL}_n(\mathcal{O})$, and contain several general results, albeit a complete classification only in certain cases. In particular, Hill's work gives a construction (up to a description of orbits) of the so called *regular* representations (resp. *split regular* in the odd conductor case).

Despite all these works, the literature on the representations of $\mathrm{GL}_2(\mathcal{O})$ remains in a somewhat unsatisfactory state: The Weil representation approach of Nobs and Wolfart works only for GL_2 and closely related groups of small rank, and Nagornyj's paper, in addition to restricting to the case $\mathcal{O} = \mathbb{Z}_p$ with p odd, omits certain details and uses an ad hoc argument in the cuspidal case. Moreover, although Hill's construction of regular representations covers most of the representations of $\mathrm{GL}_2(\mathcal{O})$, it omits a discussion of the orbits, and does not work for cuspidal representations in the odd conductor case. Hill does have a construction of so called strongly semisimple representations which include the cuspidals in the odd conductor case, but this involves a certain implicit step (see Subsect. 4.2), and a more direct construction of the cuspidal representations is desirable.

The purpose of this paper is to give a complete and uniform construction of all the irreducible representations of the groups $\mathrm{GL}_2(\mathcal{O}/\mathfrak{p}^r)$, for $r \geq 2$ (the case $r = 1$ being well-known classically), via the approach of orbits and Clifford theory. This amounts to a reconstruction of Kutzko's unpublished results, using whenever

possible, the general results of Hill in order to fit the present construction into a more general picture.

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2. ORBITS AND CLIFFORD THEORY

Let F be a local field with finite residue field k of arbitrary characteristic. Denote by \mathcal{O} its ring of integers, by \mathfrak{p} its maximal ideal, and by ϖ a prime element. For any integer $r \geq 1$ we write \mathcal{O}_r for the finite ring $\mathcal{O}/\mathfrak{p}^r$. Let $G_r = \mathrm{GL}_2(\mathcal{O}_r)$, $K_i = \{g \in G_r \mid g \equiv 1 \pmod{\mathfrak{p}^i}\}$, for $1 \leq i \leq r-1$. From now on let $i \geq r/2$; then $x \mapsto 1 + \varpi^i x$ induces an isomorphism

$$M_2(\mathcal{O}_{r-i}) \xrightarrow{\sim} K_i.$$

The group G_r acts on $M_2(\mathcal{O}_{r-i})$ by conjugation, via its quotient G_{r-i} . This action is transformed by the above isomorphism into the action of G_r on the normal subgroup K_i .

Fix an additive character $\psi : \mathcal{O} \rightarrow \mathbb{C}^\times$ with conductor \mathfrak{p}^r and define for any $\beta \in M_2(\mathcal{O}_{r-i})$, a character $\psi_\beta : K_i \rightarrow \mathbb{C}^\times$ by

$$\psi_\beta(x) = \psi(\mathrm{Tr}(\beta(x-1))).$$

Then $\beta \mapsto \psi_\beta$ gives an isomorphism

$$M_2(\mathcal{O}_{r-i}) \cong \mathrm{Hom}(K_i, \mathbb{C}^\times).$$

For $g \in G_r$ we have

$$\psi_{g\beta g^{-1}}(x) = \psi(\mathrm{Tr}(g\beta g^{-1}(x-1))) = \psi(\mathrm{Tr}(\beta g^{-1}(x-1)g)) = \psi_\beta(g^{-1}xg).$$

Thus, the above isomorphism transforms the action of G_r on $M_2(\mathcal{O}_{r-i})$ into (the inverse) conjugation of characters.

We will make use of the following well-known results of Clifford theory.

Theorem 2.1. *Let G be a finite group, and N a normal subgroup. For any irreducible representation ρ of N , define the stabilizer, $T(\rho) = \{g \in G \mid \rho^g \cong \rho\}$, of ρ . Then the following hold.*

- i) *If π is an irreducible representation of G , then $\pi|_N = e(\bigoplus_{\rho \in \Omega} \rho)$, where Ω is an orbit of irreducible representations of N under the action of G by conjugation, and e is a positive integer.*
- ii) *Suppose that ρ is an irreducible representation of N , and let*

$$A = \{\theta \in \mathrm{Irr}(T(\rho)) \mid \langle \mathrm{Res}_N^{T(\rho)}(\theta), \rho \rangle \neq 0\},$$

$$B = \{\pi \in \mathrm{Irr}(G) \mid \langle \mathrm{Res}_N^G(\pi), \rho \rangle \neq 0\}.$$

Then $\theta \mapsto \mathrm{Ind}_{T(\rho)}^G(\theta)$ is a bijection of A onto B .

- iii) *Let H be a subgroup of G containing N , and suppose that ρ is an irreducible representation of N which has an extension $\tilde{\rho}$ to H (i.e. $\tilde{\rho}|_N = \rho$). Then*

$$\mathrm{Ind}_N^H(\rho) = \bigoplus_{\chi \in \mathrm{Irr}(H/N)} \chi \tilde{\rho},$$

where each $\chi\tilde{\rho}$ is irreducible, and where we have identified $\mathrm{Irr}(H/N)$ with representations of H that are trivial on N .

iv) If ρ is an irreducible representation of N and $T(\rho)/N$ is cyclic, then there exists an extension of ρ to $T(\rho)$.

For proofs of the above, see for example [8], 6.2, 6.11, 6.17, and 11.22, respectively. The above results *i*) and *ii*) show that in order to obtain a classification of the representations of G_r , it is enough to classify the orbits of irreducible representations ψ_β of a normal subgroup K_i , and to describe all the irreducible representations of the stabilizers $T(\psi_\beta)$ which contain ψ_β (when restricted to K_i), that is, to decompose $\mathrm{Ind}_{K_i}^{T(\psi_\beta)}(\psi_\beta)$ into irreducible representations. This is what we will do in the following.

2.1. Orbits. Set $l = [\frac{r+1}{2}]$, $l' = [\frac{r}{2}]$; thus $l + l' = r$. Then K_l is the largest abelian group among the kernels K_i , so we can describe its irreducible representations ψ_β , and at the same time it has the advantage of being sufficiently close to the stabilizers of its representations to enable us to describe the irreducible components of $\mathrm{Ind}_{K_i}^{T(\psi_\beta)}(\psi_\beta)$.

We will consider G_r -orbits on $M_2(\mathcal{O}_{l'}) = M_2(\mathcal{O}_{r-l}) \cong \mathrm{Hom}(K_l, \mathbb{C}^\times)$. The Jordan and rational canonical forms imply that the orbits fall into four types, according to their reductions mod \mathfrak{p} . The following is a list of the possible orbits in $M_2(k)$:

- (1) $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$,
- (2) $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, where $a \neq d$,
- (3) $\begin{pmatrix} 0 & 1 \\ -\Delta & s \end{pmatrix}$, where $x^2 - sx + \Delta$ is irreducible,
- (4) $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$.

If $x \in M_2(\mathcal{O}_i)$, for some $i \geq 1$, we shall write \bar{x} for the image of x in $M_2(k)$. Let $\beta \in M_2(\mathcal{O}_{l'})$. Then $\psi_\beta|_{K_{r-1}} = \psi_{\bar{\beta}}$, and so if β' is another matrix in $M_2(\mathcal{O}_{l'})$, we have

$$\beta \equiv \beta' \pmod{\mathfrak{p}} \iff \psi_\beta|_{K_{r-1}} = \psi_{\beta'}|_{K_{r-1}}.$$

If $aI_2 \in M_2(k)$ is a scalar matrix (I_2 is the unit matrix), then $\psi_{aI_2}(x) = \psi(a \mathrm{Tr}(x - 1)) = \psi(a(\det(x) - 1))$. Thus $\psi_{aI_2} = \chi_a \circ \det$, for some character χ_a of the group $\{g \in \mathcal{O}_r^\times \mid g \equiv 1 \pmod{\mathfrak{p}^{r-1}}\}$. Since \mathcal{O}_r^\times is abelian, χ_a has an extension to \mathcal{O}_r^\times . Consequently ψ_{aI_2} has an extension to G_r , which we denote by $\tilde{\psi}_{aI_2}$.

Suppose now that π is a representation of G_r such that $\pi|_{K_l}$ contains ψ_β , with β a matrix of type (1) such that $\bar{\beta} = aI_2$. Then $(\pi \otimes \tilde{\psi}_{-aI_2})|_{K_{r-1}} = \psi_0|_{K_{r-1}} = \mathbf{1}_{K_{r-1}}$. Thus π factors through G_{r-1} after twisting by a one-dimensional character, i.e. π has the group $K_{r-1} \cap \mathrm{SL}_2(\mathcal{O}_r)$ in its kernel. Working inductively, we assume that the representations of G_{r-1} are known. Hence the representations corresponding to orbits of type (1) are described.

Similarly, if π is a representation of G_r corresponding to an orbit of type (4) containing an element β such that $\bar{\beta} = aI_2 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $\pi \otimes \tilde{\psi}_{-aI_2}$ corresponds to an orbit in $M_2(\mathcal{O}_{l'})$ that can be represented by a matrix β' such that $\beta' \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(mod \mathfrak{p}). This implies that $\beta' = \begin{pmatrix} a & 1+b \\ c & d \end{pmatrix}$ for some $a, b, c, d \in \mathfrak{p}$. Conjugating β' by $\begin{pmatrix} 1 & 0 \\ a & 1+b \end{pmatrix}$, we can assume that β' has the form $\begin{pmatrix} 0 & 1 \\ -\Delta & s \end{pmatrix}$ for some $\Delta, s \in \mathfrak{p}$.

If β represents an orbit of type (3), such that $\beta \equiv \begin{pmatrix} 0 & 1 \\ -\Delta & s \end{pmatrix} \pmod{\mathfrak{p}}$, then $\beta = \begin{pmatrix} a & 1+b \\ -\Delta+c & s+d \end{pmatrix}$ for some $a, b, c, d \in \mathfrak{p}$, and conjugating by $\begin{pmatrix} 1 & 0 \\ a & 1+b \end{pmatrix}$, we can assume that β has the form $\begin{pmatrix} 0 & 1 \\ -\Delta & s \end{pmatrix}$ for some $\Delta, s \in \mathcal{O}_{l'}$.

We have therefore reduced the problem of describing the representations of G_r to describing the representations corresponding to orbits in $M_2(\mathcal{O}_{l'})$, represented by elements of the following types

- (1') $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, with $a \not\equiv d \pmod{\mathfrak{p}}$,
- (2') $\begin{pmatrix} 0 & 1 \\ -\Delta & s \end{pmatrix}$, where $x^2 - sx + \Delta$ is irreducible mod \mathfrak{p} ,
- (3') $\begin{pmatrix} 0 & 1 \\ -\Delta & s \end{pmatrix}$, where $\Delta, s \in \mathfrak{p}$.

All of these orbits are *regular* in the sense of Hill [6]. Indeed, by Theorem 3.6 in [6], an element $\beta \in M_2(\mathcal{O}_{l'})$ is regular if and only if the centralizer $C_{\mathbf{G}_1}(\bar{\beta})$ is abelian, where $\mathbf{G}_1 = \mathrm{GL}_2(\bar{k})$, and \bar{k} is an algebraic closure of k . Now simple calculations show that $C_{\mathbf{G}_1}(\bar{\beta})$ is conjugate to the group

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \left\{ \begin{pmatrix} a & b \\ -\bar{\Delta}b & a+\bar{s}b \end{pmatrix} \in \mathrm{GL}_2(\bar{k}) \right\}, \text{ or } \left\{ \begin{pmatrix} a & * \\ 0 & a \end{pmatrix} \mid a \in k^\times \right\},$$

when β is of type (1'), (2'), or (3'), respectively.

We observe that the representations corresponding to orbits of type (1') and (3') are *split regular* in the sense of Hill (cf. [6], 4.4). This means that the orbits are regular and that the characteristic polynomial splits completely mod \mathfrak{p} (not necessarily into distinct factors). The representations corresponding to orbits of type (2') are *cuspidal* (cf. [7], 4.2), i.e. the characteristic polynomial is irreducible mod \mathfrak{p} .

Let $\beta \in M_2(\mathcal{O}_{l'})$ be an element in an orbit of one of the above three types. Since each of these orbits is regular, it follows from Corollary 3.7 in [6] that the stabilizer of ψ_β in G_r is given by

$$T(\psi_\beta) = \mathcal{O}_r[\hat{\beta}]^\times K_{l'},$$

where $\hat{\beta}$ is a lift of β to an element in $M_2(\mathcal{O}_r)$.

To describe the representations of G_r corresponding to the above orbits, we shall proceed by considering two cases depending on whether r is even or odd.

3. THE CASE $r = 2l$, $l' = l$

Let β be a representative of one of the three orbits above, and choose a lift $\hat{\beta} \in M_2(\mathcal{O}_r)$. Let $\theta \in \mathrm{Hom}(\mathcal{O}_r[\hat{\beta}]^\times, \mathbb{C}^\times)$ be a character such that θ is equal to ψ_β when restricted to $K_l \cap \mathcal{O}_r[\hat{\beta}]^\times$ (such a θ exists since $\mathcal{O}_r[\hat{\beta}]^\times$ is abelian). Then $\theta\psi_\beta$ is a one-dimensional representation of $T(\psi_\beta)$ such that $\theta\psi_\beta(xy) = \theta(x)\psi_\beta(y)$, for $x \in \mathcal{O}_r[\hat{\beta}]^\times$, $y \in K_l$.

Define the representation

$$\pi(\theta, \beta) = \mathrm{Ind}_{\mathcal{O}_r[\hat{\beta}]^\times K_l}^{G_r}(\theta\psi_\beta).$$

By Theorem 2.1 ii), $\pi(\theta, \beta)$ is an irreducible representation, $\pi(\theta, \beta) = \pi(\theta', \beta) \Rightarrow \theta = \theta'$, and every irreducible representation of G_r corresponding to an orbit of one

of the above three types is obtained in this way. This is a special case of Theorem 4.1 in [6], which holds for any GL_n and any regular orbit.

We observe that a different choice of representative β for the orbit, gives a conjugate character ψ_β and a conjugate stabilizer $T(\psi_\beta)$. The resulting induced representations are therefore isomorphic for any choice of representative β . Moreover, a different choice of $\hat{\beta}$, even though it may give a different group $\mathcal{O}_r[\hat{\beta}]^\times$ and different characters θ , will give a resulting set of representations $\theta\psi_\beta$, that equals the original one. This is because in both cases we obtain the set of irreducible components of $\mathrm{Ind}_{K_l}^{T(\psi_\beta)}(\psi_\beta)$. Thus, the set of representations $\pi(\theta, \beta)$ depends only on the orbit of β , and is independent of the choice of lift $\hat{\beta}$.

4. THE CASE $r = 2l - 1$, $l' = l - 1$

As before, Let β be a representative of one of the three orbits above, and choose a lift $\hat{\beta} \in M_2(\mathcal{O}_r)$. The thing that makes this case more difficult is that we start from a character ψ_β on K_l , while $T(\psi_\beta) = \mathcal{O}_r[\hat{\beta}]^\times K_{l'}$, and $K_l \not\leq K_{l'}$, so it is not as easy to decompose $\mathrm{Ind}_{K_l}^{T(\psi_\beta)}(\psi_\beta)$. This the “odd conductor case” in turn falls into two subcases: the split case, and the cuspidal case.

4.1. The split case. Following Hill [6], Sect. 4, we assume that β lies in a split regular orbit, i.e. of type (1') or (3'). Let B_r be the subgroup of G_r of upper-triangular matrices. By the proof of Lemma 4.5 in [6], the group $H_\beta = K_l(B_r \cap K_{l'})$ has the property that H_β/K_l is a maximal isotropic subspace of $K_{l'}/K_l$ with respect to the form¹

$$\langle \cdot, \cdot \rangle_{\overline{\beta}} : K_{l'}/K_l \times K_{l'}/K_l \longrightarrow k, \quad \langle xK_l, yK_l \rangle_{\overline{\beta}} = \mathrm{Tr}(\overline{\beta}(\overline{mn} - \overline{nm})),$$

where $x = 1 + \varpi^{l'}m$ and $y = 1 + \varpi^{l'}n$. Moreover, H_β is normal in $T(\psi_\beta)$, and we have the following diagram of groups.

$$\begin{array}{ccc} & & T(\psi_\beta) \\ & \nearrow & \downarrow \\ K_{l'} & & \mathcal{O}_r[\hat{\beta}]^\times H_\beta \\ \downarrow & \nearrow & \downarrow \\ H_\beta & & \mathcal{O}_r[\hat{\beta}]^\times K_l \\ \downarrow & \nearrow & \\ K_l & & \end{array}$$

We now review the main steps of the proof of Theorem 4.6 in [6]. Let $N = \ker(\psi_\beta)$ and write $\overline{\psi}_\beta$ for ψ_β viewed as a character of K_l/N . One can show that H_β/N and $\mathcal{O}_r[\hat{\beta}]^\times K_l/N$ are both abelian. Hence, $\overline{\psi}_\beta$ can be extended to both these groups. Next it is shown that there exists exactly q^2 (q^n in the case of GL_n) $\mathcal{O}_r[\hat{\beta}]^\times H_\beta/N$ -stable extensions of $\overline{\psi}_\beta$ to H_β/N . Let ψ'_β be one of these. Furthermore, one can

¹The form takes values in k , not in \mathbb{C} as stated by Hill.

construct an extension ψ''_β of $\overline{\psi}_\beta$ to $\mathcal{O}_r[\hat{\beta}]^\times K_l/N$, such that ψ''_β and ψ'_β agree on $(H_\beta/N) \cap (\mathcal{O}_r[\hat{\beta}]^\times K_l/N)$. Then one can check that the map

$$\psi'''_\beta : \mathcal{O}_r[\hat{\beta}]^\times H_\beta/N \longrightarrow \mathbb{C}, \quad \psi'''_\beta(xhN) = \psi''_\beta(xN)\psi'_\beta(hN),$$

where $x \in \mathcal{O}_r[\hat{\beta}]^\times, h \in H_\beta$, is a well-defined homomorphism. Denote by $\tilde{\psi}_\beta$ the composition of ψ'''_β with the natural homomorphism $\mathcal{O}_r[\hat{\beta}]^\times H_\beta \rightarrow \mathcal{O}_r[\hat{\beta}]^\times H_\beta/N$. Thus $\tilde{\psi}_\beta$ is an extension of ψ_β to $\mathcal{O}_r[\hat{\beta}]^\times H_\beta$.

Now define the characters

$$\rho_\beta = \mathrm{Ind}_{H_\beta}^{K_{l'}}(\tilde{\psi}_\beta|_{H_\beta}), \text{ and } \zeta_\beta = \mathrm{Ind}_{\mathcal{O}_r[\hat{\beta}]^\times H_\beta}^{T(\psi_\beta)}(\tilde{\psi}_\beta).$$

It follows from [6], Prop. 4.2, that ρ_β is irreducible. Moreover, it is shown that ζ_β is an extension of ρ_β . Frobenius reciprocity and counting degrees shows that the induced representation $\mathrm{Ind}_{K_{l'}}^{T(\psi_\beta)}(\rho_\beta)$ decomposes as the sum of the representations $\omega\zeta_\beta$, where ω runs through the linear characters of the abelian group $T(\psi_\beta)/K_{l'}$. Finally, it is shown that every irreducible component of $\mathrm{Ind}_{K_{l'}}^{K_{l'}}(\psi_\beta)$ has the form $\mathrm{Ind}_{H_\beta}^{K_{l'}}(\psi'_\beta)$ for some $\mathcal{O}_r[\hat{\beta}]^\times$ -stable extension ψ'_β of ψ_β . Thus, every irreducible component of $\mathrm{Ind}_{K_{l'}}^{T(\psi_\beta)}(\psi_\beta)$ has the form $\omega\zeta_\beta$, where ζ_β is induced from some $\tilde{\psi}_\beta$, and ω is as above, and all the characters $\omega\zeta_\beta$ are distinct for distinct ψ'_β and ω .

Thus, every representation $\omega\zeta_\beta$ is isomorphic to an induced representation

$$\mathrm{Ind}_{\mathcal{O}_r[\hat{\beta}]^\times H_\beta}^{T(\psi_\beta)}(\omega'\tilde{\psi}_\beta),$$

where ω' is a linear character of $\mathcal{O}_r[\hat{\beta}]^\times H_\beta/H_\beta$, and distinct ω' give distinct $\omega\zeta_\beta$.

4.2. The cuspidal case. We now assume that β is in a cuspidal orbit, i.e. of type (2'). In [7], Prop. 3.6 Hill gives a construction of so called strongly semisimple representations, which include the cuspidal ones as a special case. This construction is less explicit than ours because it involves the non-constructive existence of an irreducible constituent of $\mathrm{Ind}_{K_{l'}}^{K_{l'}}(\psi_\beta)$. For the construction of the cuspidal representations, there are also several slightly different methods giving parametrizations in terms of certain characters of $\mathcal{O}_{F_2}^\times$, where F_2 is the quadratic unramified extension of F (cf. [1] and its references). The method we give here is similar to a more general construction used by Bushnell and Kutzko (cf. [3]) in their description of supercuspidal representations of $\mathrm{GL}_n(F)$. In particular, the method can be adapted to the construction of cuspidal representations for any $\mathrm{GL}_n(\mathcal{O})$. We will make use of the following result from [2], 8.3.3.

Proposition 4.1. *Let G be a finite group and N a normal subgroup, such that G/N is an elementary abelian p -group. Thus G/N has a structure of \mathbb{F}_p -vector space. Let χ be a one-dimensional representation of N , which is stabilized by G . Define a alternating bilinear form*

$$h_\chi : G/N \times G/N \longrightarrow \mathbb{C}^\times, \quad h_\chi(g_1N, g_2N) = \chi([g_1, g_2]) = \chi(g_1g_2g_1^{-1}g_2^{-1}).$$

Assume that the form h_χ is non-degenerate. Then there exists a unique up to isomorphism irreducible representation ρ_χ of G such that $\rho_\chi|_N$ contains χ .

Note that the form is well-defined², so the non-trivial thing here is the uniqueness part.

Set $Z^1 = \mathcal{O}_r[\hat{\beta}]^\times \cap K_1$. The strategy to find the irreducible constituents of $\mathrm{Ind}_{K_l}^{T(\psi_\beta)}(\psi_\beta)$ is as follows. First we extend ψ_β to a representation $\tilde{\psi}_\beta$ of $Z^1 K_l$, as in the even conductor case. Next we apply the above proposition to the case where $G = Z^1 K_{l'}$, $N = Z^1 K_l$, and $\chi = \tilde{\psi}_\beta$. We thus obtain a representation η_β containing ψ_β , and it is clear that every representation of $Z^1 K_{l'}$ containing ψ_β is of this form. Finally, we use Theorem 2.1, *iv*) to lift η_β to a representation of $T(\psi_\beta)$. Again it is clear that we have in this way obtained every irreducible component of $\mathrm{Ind}_{K_l}^{T(\psi_\beta)}(\psi_\beta)$.

To carry out the above strategy, we need to establish several facts concerning the groups $T(\psi_\beta)$, $Z^1 K_{l'}$, $Z^1 K_l$, and their respective representations: First, to apply Proposition 4.1, we need show that $Z^1 K_l$ is normal in $Z^1 K_{l'}$, that the quotient is elementary abelian, that $Z^1 K_{l'}$ stabilizes the representation $\tilde{\psi}_\beta$, and that the form $h_{\tilde{\psi}_\beta}$ is non-degenerate. Then, to apply Theorem 2.1, *iv*), we need to show that $Z^1 K_{l'}$ is normal in $T(\psi_\beta)$, that $T(\psi_\beta)$ stabilizes the representation η_β , and that the quotient $T(\psi_\beta)/Z^1 K_{l'}$ is cyclic.

We prove all the above facts in one go. First we show that both $Z^1 K_{l'}$ and $Z^1 K_l$ are normal in $T(\psi_\beta)$ (and thus $Z^1 K_l$ is normal in $Z^1 K_{l'}$). It is clear that $Z^1 K_{l'}$ is normal in $T(\psi_\beta)$ since $Z^1 K_{l'} = T(\psi_\beta) \cap K_1$, and K_1 is normal in G_r . Now let $g \in T(\psi_\beta)$ and $x \in Z^1 K_l$. Write $g = ek$ with $e \in \mathcal{O}_r[\hat{\beta}]^\times$ and $k \in K_{l'}$, and write $x = zh$, with $z \in Z^1$ and $h \in K_l$. Then

$$gxg^{-1} = z \cdot e([z^{-1}, k](khk^{-1}))e^{-1},$$

and the two factors in the bracket lie in K_l since $[K_1, K_{l'}] \leq K_l$. Since K_l is normal in G_r , the whole expression $e([z^{-1}, k](khk^{-1}))e^{-1}$ in fact lies in K_l , so $gxg^{-1} \in Z^1 K_l$, as required. Now

$$\begin{aligned} \tilde{\psi}_\beta(gxg^{-1}) &= \tilde{\psi}_\beta(z)\psi_\beta(e([z^{-1}, k](khk^{-1}))e^{-1}) \\ &= \tilde{\psi}_\beta(z)\psi_\beta([z^{-1}, k])\psi_\beta(h) \\ &= \tilde{\psi}_\beta(x)\psi_\beta([z^{-1}, k]), \end{aligned}$$

where the second equality follows since $T(\psi_\beta)$ stabilizes ψ_β . Thus, to show that $T(\psi_\beta)$ stabilizes $\tilde{\psi}_\beta$, we need only to show that $\psi_\beta([z^{-1}, k]) = 1$. For this, write $k = 1 + y$, with $y \in \mathfrak{p}' M_2(\mathcal{O}_r)$ and notice that

$$[z^{-1}, k] = z(1 + y)z^{-1}(1 - y + y^2) = z(1 + y)z^{-1} - y,$$

(here we have used the fact that $y^3 = 0$ and $zy^2 = y^2$ in $M_2(\mathcal{O}_r)$). Then

$$\psi_\beta([z^{-1}, k]) = \psi(\mathrm{Tr}(\beta(zyz^{-1} - y))) = \psi(\mathrm{Tr}(z(\beta y)z^{-1} - \beta y)) = 1,$$

where we have used $z\hat{\beta} = \hat{\beta}z \Rightarrow z\beta = \beta z$, and the fact that Tr is a conjugacy class function.

² $\chi([g_1 n_1, g_1 n_2]) = \chi(g_1 g_2 (g_2^{-1} n_1 g_2 n_2 n_1^{-1} g_1^{-1} n_2^{-1} g_1) g_1^{-1} g_2^{-1})$
 $= \chi(g_1 g_2 (g_2^{-1} n_1 g_2 n_2 n_1^{-1} g_1^{-1} n_2^{-1} g_1) (g_1 g_2)^{-1} [g_1, g_2])$
 $= \chi(g_2^{-1} n_1 g_2 n_2 n_1^{-1} g_1^{-1} n_2^{-1} g_1) \chi([g_1, g_2]) = \chi(g_2^{-1} n_1 g_2) \chi(n_2 n_1^{-1}) \chi(g_1^{-1} n_2^{-1} g_1) \chi([g_1, g_2])$
 $= \chi([g_1, g_2]).$

Next, note that since $[K_{l'}, K_{l'}] \leq K_l$, the bilinear form does not depend on the lift $\tilde{\psi}_\beta$. The form $h_{\psi_\beta} = h_{\tilde{\psi}_\beta}$ is related to the above form $\langle \cdot, \cdot \rangle_{\bar{\beta}}$ in the following way. First, there is a natural isomorphism

$$\frac{K_{l'}}{(Z^1 \cap K_{l'})K_l} \xrightarrow{\sim} \frac{Z^1 K_{l'}}{Z^1 K_l}, \quad k(Z^1 \cap K_{l'})K_l \mapsto kZ^1 K_l, \text{ where } k \in K_{l'},$$

and moreover, it is shown in the proof of Prop. 4.2 of [6] that $\langle xK_l, yK_l \rangle_{\bar{\beta}} = 0$ if and only if $\psi_\beta([x, y]) = 1$, so $h_{\psi_\beta}(xZ^1 K_l, yZ^1 K_l) = 0$ if and only if $\langle xK_l, yK_l \rangle_{\bar{\beta}} = 0$. By [6], Corollary 4.3³ the radical of the form $\langle \cdot, \cdot \rangle_{\bar{\beta}}$ is equal to $(\mathcal{O}_r[\hat{\beta}]^\times \cap K_{l'})K_l/K_l$, so by the above isomorphism the radical of h_{ψ_β} is $Z^1 K_l(\mathcal{O}_r[\hat{\beta}]^\times \cap K_{l'})/Z^1 K_l = Z^1 K_l/Z^1 K_l = 1$. We thus conclude that the form h_{ψ_β} is non-degenerate. Since $K_{l'}/K_l \cong M_2(k)$ is elementary abelian, the same is true for any of its quotients.

Finally, we show that $T(\psi_\beta)/Z^1 K_{l'}$ is cyclic, and that $T(\psi_\beta)$ stabilizes the representation η_β . The former is true because the quotient is isomorphic to $\mathcal{O}_r[\hat{\beta}]^\times/Z^1 \cong k[\bar{\beta}]^\times$, and this is the multiplicative group of a finite field (the residue field of the unramified extension generated by a lift of β to $M_2(\mathcal{O}_r)$). The latter is true because for $g \in T(\psi_\beta)$, the conjugate representation η_β^g is another irreducible representation of $Z^1 K_{l'}$ whose restriction to $Z^1 K_l$ contains $\tilde{\psi}_\beta^g = \tilde{\psi}_\beta$, so the uniqueness in Prop. 4.1 implies that η_β^g is equivalent to η_β .

To conclude, we have shown that any irreducible component of $\mathrm{Ind}_{K_l}^{T(\psi_\beta)}(\psi_\beta)$ is of the form $\tilde{\eta}_\beta$, for some lift $\tilde{\eta}_\beta$ of η_β to $T(\psi_\beta)$. It follows that any irreducible representation of G_r containing ψ_β is of the form

$$\mathrm{Ind}_{T(\psi_\beta)}^{G_r}(\tilde{\eta}_\beta),$$

for some $\tilde{\eta}_\beta$, and distinct inducing representations give distinct induced representations. Again the dependence on the choice of lift $\hat{\beta}$ is irrelevant, since for any choice we obtain the set of irreducible components of $\mathrm{Ind}_{K_l}^{T(\psi_\beta)}(\psi_\beta)$. The choice of representative β is irrelevant for the same reason as before.

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³Note that the second equality in the corollary is incorrect.

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